ON THE STABILITY OF A GYROSCOPIC SYSTEM

(OB USTOICHIVOSTI ODNOI GIROSKOPICHESKOI SISTEMY)

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When using the second method of Liapunov in investigating the problem of stability of motion of a gyroscopic stabilizing system installed on a ship performing complex manoeuvers, it is necessary to construct Liapunov's function for variational equations. These equations in this case will constitute a system of linear differential equations with variable coefficients. Successful methods of constructing Liapunov functions for such systems are given by Chetaev [1]. Boundaries of the stable regions obtained by means of these methods are investigated by Razumikhin [2]. In the paper by Roitenberg [3] (closely related to the two mentioned above) the Liapunov function is constructed by first transforming the original system of differential equations to new variables which are coordinates of some auxiliary system of differential equations. Utilizing this method, let us construct the Liapunov function for the differential equations of the gyroscopic system under consideration having the following form:

$$x_1 + x_2^{\cdot} - f_1(t)\, x_3 = 0, \qquad x_1^{\cdot} - m_1 x_2 + f_1(t)\, x_4 - m_2 x_5 = 0$$

$$-f_1(t)\, x_1 + m_1 x_3 + x_4^{\cdot} + m_3 x_4 = 0, \qquad f_1(t)\, x_2 + x_3^{\cdot} - [m_4 + f_2(t)]\, x_4 = 0 \qquad (1)$$

$$m_5 x_2 + x_5^{\cdot} + m_5 x_5 = 0$$

Here $x_1, \ldots, x_5$ are coordinates of the system, $m_1, \ldots, m_5$ are constant coefficients, $f_1(t)$ and $f_2(t)$ are some time functions dependent on the ship's movement.

Let us transform the system of equations (1) to the new variables ξ_i $(i = 1, \ldots, 5)$; these variables are the coordinates of the system of linear differential equations with constant coefficients obtained from the system (1) when $f_1(t) \equiv f_2(t) \equiv 0$.

For real values of the parameters, the characteristic equation of the above system of differential equations with constant coefficients shall have one real root and two pairs of complex roots which may be denoted as follows:

$$\lambda_1 = \varkappa, \qquad \lambda_2, \lambda_3 = \varepsilon_1 \pm i\omega_1, \qquad \lambda_4, \lambda_5 = \varepsilon_2 \pm i\omega_2 \tag{2}$$

Variables ξ_i are determined from the following relationships [3]:

$$x_1 = \frac{\varkappa(\varkappa + m_5)}{m_5}\xi_1 + \frac{\varepsilon_1(\varepsilon_1 + m_5) - \omega_1^2}{m_5}\xi_2 + \frac{(2\varepsilon_1 + m_5)\omega_1}{m_5}\xi_3$$
$$x_2 = -\frac{\varkappa + m_5}{m_5}\xi_1 - \frac{\varepsilon_1 + m_5}{m_5}\xi_2 - \frac{\omega_1}{m_5}\xi_3$$
$$x_3 = \xi_4, \qquad x_4 = \frac{1}{m_4}(\varepsilon_2\xi_4 + \omega_2\xi_5), \qquad x_5 = \xi_1 + \xi_2 \tag{3}$$

New variables ξ_i shall satisfy the following differential equations:

$$\frac{d\xi_1}{dt} = \varkappa\xi_1 + f_1(t)(c_{14}\xi_4 + c_{15}\xi_5)$$
$$\frac{d\xi_2}{dt} = \varepsilon_1\xi_2 + \omega_1\xi_3 + f_1(t)(c_{24}\xi_4 + c_{25}\xi_5)$$
$$\frac{d\xi_3}{dt} = \varepsilon_1\xi_3 - \omega_1\xi_2 + f_1(t)(c_{34}\xi_4 + c_{35}\xi_5) \tag{4}$$
$$\frac{d\xi_4}{dt} = \varepsilon_2\xi_4 + \omega_2\xi_5 + f_1(t)(c_{41}\xi_1 + c_{42}\xi_2 + c_{43}\xi_3) + f_2(t)(c_{44}\xi_4 + c_{45}\xi_5)$$
$$\frac{d\xi_5}{dt} = \varepsilon_2\xi_5 - \omega_2\xi_4 + f_1(t)(c_{51}\xi_1 + c_{52}\xi_2 + c_{53}\xi_3) + f_2(t)(c_{54}\xi_4 + c_{55}\xi_5)$$

where

$$c_{14} = -c_{24} = -\frac{m_5(\varkappa m_4 + \varepsilon_2)}{m_4[\omega_1^2 + (\varepsilon_1 - \varkappa)^2]}, \quad c_{15} = -c_{25} = -\frac{m_5\omega_2}{m_4[\omega_1^2 + (\varepsilon_1 - \varkappa)^2]}$$
$$c_{34} = -\frac{m_5[m_4\omega_1^2 + (\varepsilon_1 - \varkappa)(\varepsilon_1 + \varepsilon_2)]}{m_4\omega_1[\omega_1^2 + (\varepsilon_1 - \varkappa)^2]}, \qquad c_{35} = -\frac{m_5\omega_2(\varepsilon_1 - \varkappa)}{m_4\omega_1[\omega_1^2 + (\varepsilon_1 - \varkappa)^2]}$$
$$c_{41} = \frac{m_5 + \varkappa}{m_5}, \qquad c_{51} = \frac{m_5 + \varkappa}{m_5\omega_2}(\varepsilon_2 + m_3 + \varkappa m_4) \tag{5}$$
$$c_{42} = \frac{m_5 + \varepsilon_1}{m_5}, \qquad c_{52} = \frac{m_5 + \varepsilon_1}{m_5\omega_2}(\varepsilon_2 + m_3 + \varepsilon_1 m_4) - \frac{m_4\omega_1^2}{m_5\omega_2}$$
$$c_{43} = \frac{\omega_1}{m_5}, \qquad c_{53} = \frac{\omega_1}{m_5\omega_2}[\varepsilon_2 + m_3 + m_4(2\varepsilon_1 + m_5)]$$
$$c_{44} = \frac{\varepsilon_2}{m_4}, \qquad c_{54} = \frac{(\varepsilon_2 + m_3)\varepsilon_2}{m_4\omega_2}, \qquad c_{45} = \frac{\omega_2}{m_4}, \qquad c_{55} = \frac{\varepsilon_2 + m_3}{m_4}$$

Let us use the following definite negative function as Liapunov's function:

$$V = -\frac{1}{2}[p_1\xi_1^2 + p_2(\xi_2^2 + \xi_3^2) + p_3(\xi_4^2 + \xi_5^2)] \tag{6}$$

where p_1, p_2, p_3 are some positive constant coefficients. Differentiating (6) with respect to time we obtain

$$V^{\cdot} = -\left[p_1 \frac{d\xi_1}{dt}\xi_1 + p_2\left(\frac{d\xi_2}{dt}\xi_2 + \frac{d\xi_3}{dt}\xi_3\right) + p_3\left(\frac{d\xi_4}{dt}\xi_4 + \frac{d\xi_5}{dt}\xi_5\right)\right] \quad (7)$$

After substituting the values of $d\xi_i/dt (i = 1, \ldots, 5)$ from equations (4) into (7), we have

$$V^{\cdot} = a_{11}\xi_1^2 + a_{22}\xi_2^2 + a_{33}\xi_3^2 + a_{44}\xi_4^2 + a_{55}\xi_5^2 + 2a_{14}f_1(t)\,\xi_1\xi_4 + 2a_{15}f_1(t)\,\xi_1\xi_5 + $$
$$+ 2a_{24}f_1(t)\,\xi_2\xi_4 + 2a_{25}f_1(t)\,\xi_2\xi_5 + 2a_{34}f_1(t)\,\xi_3\xi_4 + 2a_{35}f_1(t)\,\xi_3\xi_5 + 2a_{45}f_2(t)\,\xi_4\xi_5 \quad (8)$$

where

$$a_{11} = -p_1\varkappa, \quad a_{22} = a_{33} = -p_2\varepsilon_1, \quad a_{44} = -p_3[\varepsilon_2 + c_{44}f_2(t)], \quad a_{55} = -p_3[\varepsilon_2 + c_{55}f_2(t)]$$

$$a_{14} = -\frac{p_1c_{14} + p_3c_{41}}{2}, \quad a_{15} = -\frac{p_1c_{15} + p_3c_{51}}{2}, \quad a_{24} = -\frac{p_2c_{24} + p_3c_{42}}{2}, \quad a_{25} = -\frac{p_2c_{25} + p_3c_{52}}{2}$$

$$a_{34} = -\frac{p_2c_{34} + p_3c_{43}}{2}, \quad a_{35} = -\frac{p_2c_{35} + p_3c_{53}}{2}, \quad a_{45} = -\frac{p_3(c_{45} + c_{54})}{2} \quad (9)$$

The discriminant of the quadratic form (8) is as follows:

$$D = \begin{vmatrix} a_{11} & 0 & 0 & a_{14}f_1(t) & a_{15}f_1(t) \\ 0 & a_{22} & 0 & a_{24}f_1(t) & a_{25}f_1(t) \\ 0 & 0 & a_{33} & a_{34}f_1(t) & a_{35}f_1(t) \\ a_{14}f_1(t) & a_{24}f_1(t) & a_{34}f_1(t) & a_{44} & a_{45}f_2(t) \\ a_{15}f_1(t) & a_{25}f_1(t) & a_{35}f_1(t) & a_{45}f_2(t) & a_{55} \end{vmatrix} \quad (10)$$

The sufficient conditions of asymptotic stability for the system (1) are constituted by the requirement that $V^{\cdot}$ must be a positive definite quadratic form. The sufficient and necessary condition for this requirement to be satisfied is constituted by the requirement that all diagonal minors of $D_i (i = 1, \ldots, 5)$ of the discriminant (10) be always positive for any value of time t, i.e. that for any t the following inequalities be satisfied

$$D_1 = -p_1\varkappa > 0, \qquad D_2 = p_1p_2\varkappa\varepsilon_1 > 0, \qquad D_3 = -p_1p_2^2\varkappa\varepsilon_1^2 > 0$$

$$D_4 = p_1p_2^2\varkappa\varepsilon_1^2a_{44} - p_2[p_2a_{14}^2 + p_1\varepsilon_1\varkappa(a_{24}^2 + a_{34}^2)]f_1^2 > 0$$

$$D_5 = \{-p_1\varkappa(a_{24}a_{35} - a_{25}a_{34})^2 - p_2\varepsilon_1[(a_{14}a_{35} - a_{15}a_{34})^2 + (a_{14}a_{25} - a_{15}a_{24})^2]\}f_1^4 -$$
$$- p_2\{a_{44}[p_1\varkappa\varepsilon_1(a_{35}^2 + a_{25}^2) + p_2\varepsilon_1^2a_{15}^2] + a_{55}[p_1\varkappa\varepsilon_1(a_{34}^2 + a_{24}^2) + p_2\varepsilon_1^2a_{14}^2] -$$
$$- 2[p_1\varkappa\varepsilon_1(a_{34}a_{35} + a_{24}a_{25}) + p_2\varepsilon_1^2a_{14}a_{15}]a_{45}f_2\}f_1^2 - p_1p_2^2\varkappa\varepsilon_1^2(a_{44}a_{55} - a_{45}^2f_2^2) > 0$$

Let us construct the stability region in the f_1f_2-plane. In order to maximize this region, let us vary values of the coefficients p_1, p_2, and p_3 in the Liapunov function (6). For the following values of the parameters of the system (1)

$$m_1 = 1.54 \times 10^{-6}\ \text{sec}^2, \quad m_2 = 0.924 \times 10^{-6}\ \text{sec}^2, \quad m_3 = 6 \times 10^{-3}\ \text{sec}^{-1},$$
$$m_4 = 25.974, \quad m_5 = 10^{-3}\ \text{sec}^{-1}$$

the optimum values of the variable coefficients are as follows:

$$p_1 = 0.1, \qquad p_2 = 1, \qquad p_3 = 0.127$$

The inequalities (11) are satisfied in the following region

$$|f_1(t)| \leqslant 0.75 \cdot 10^{-3} \sec^{-1}, \qquad |f_2(t)| \leqslant 2 \qquad (12)$$

These inequalities (12) define a region inside which the functions $f_1(t)$ and $f_2(t)$ may vary arbitrarily without introducing instability into the gyroscopic system under consideration.

It follows from the Hurwitz criterion that for f_1 = const, f_2 = const, the system will be stable in the region

$$|f_1| \leqslant 1.24 \cdot 10^{-3} \sec^{-1}, \qquad |f_2| \leqslant 2$$

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